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Functions with Strictly Decreasing Distances from Increasing Tchebycheff Subspaces

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Let $\{u_i\}_{i=0}^{\infty}$ be a sequence of continuous functions on [0, 1] such that $(u_0, ..., u_k)$ is a Tchebycheff system on [0, 1] for all $k \geqslant 0$ and let $C(u_0, ..., u_k)$ denote the corresponding generalized convexity cone. It is proved that if f belongs to $C(u_0, ..., u_{n-1})$, then its distance from the linear space spanned by $(u_0, ..., u_n)$ is strictly smaller than its distance from the linear space spanned by $(u_0, ..., u_{n-1})$. Other properties of the best approximants to such functions are also given.

It is shown, by a general category argument, that no direct converse can exit. It is then established that if strict decrease of distances (or one of a number of other properties of the best approximants) holds for *all* subintervals of [0, 1], then $f \in C(u_0, ..., u_{n-1})$ for all of these.

I. DIRECT THEOREMS

Let $\{u_i\}_0^{\infty}$ be an infinite sequence of continuous functions on [0, 1] such that for all n, $n \geqslant 1$, $(u_0, ..., u_{n-1})$ is a positive Tchebycheff system (T-system), i.e.,

$$\begin{vmatrix} u_0(x_1) & \cdots & u_0(x_n) \\ u_1(x_1) & \cdots & u_1(x_n) \\ \vdots & & \vdots \\ u_{n-1}(x_1) & \cdots & u_{n-1}(x_n) \end{vmatrix} > 0 \tag{1.1}$$

for all $0 \leqslant x_1 < x_2 < \cdots < x_n \leqslant 1$.

DEFINITION 1. A function f for which

$$\begin{vmatrix} u_0(x_1) & \cdots & u_0(x_{n+1}) \\ \vdots & & \vdots \\ u_{n-1}(x_1) & \cdots & u_{n-1}(x_{n+1}) \\ f(x_1) & \cdots & f(x_{n+1}) \end{vmatrix} \geqslant 0 \tag{1.2}$$

for all $0 \leqslant x_1 < x_2 < \cdots < x_{n+1} \leqslant 1$, is said to be convex with respect to $(u_0, \ldots, u_{n-1})$. The cone of these functions is denoted by $C(u_0, \ldots, u_{n-1})$ and is called a *generalized convexity cone.*

Properties of such cones have been recently investigated in several papers [1, 7, 8], and in the monograph [5].

Notation. We denote by $\Lambda(u_0, \ldots, u_{n-1})$ the n-dimensional linear space spanned by $(u_0, \ldots, u_{n-1})$. When no ambiguity exists, we abbreviate this to Λ_{n-1}.

We further denote by

$$T_{n-1}([a, b]; f) = T_{n-1}(f)$$

the best approximant, in the uniform norm on $[a, b]$, from Λ_{n-1} to f. There exists a unique best approximant since $\{u_i\}_0^{n-1}$ was assumed to be a T-system (see, e.g., [3]).

Finally, we let

$$E_{n-1}(f) = E_{n-1}([a, b]; f) = \|f - T_{n-1}([a, b]; f)\|$$

be the distance from Λ_{n-1} to f (in the uniform norm, which is the only one we use).

DEFINITION 2. Let $g(x)$ be a continuous function on $[a, b]$. A point x for which $|g(x)| = \|g\|$ is called an *extremal point* for g. An extremal point x is called a $(+)$ point if $g(x) > 0$; otherwise it is called a $(-)$ point. A sequence of extremal points for g which are $(+)$ points and $(-)$ points alternatingly, is called an "alternance" of g.

In this section we shall establish properties of the best approximants from Λ_{n-1} to generalized convex functions. We take as $[a, b]$ the fixed interval $[0, 1]$.

THEOREM 1. *Let f be a function belonging to $C(u_0, \ldots, u_{n-1}) \setminus \Lambda_{n-1}$. Then*

$$E_n(f) < E_{n-1}(f). \tag{1.3}$$

Proof. Let $P \in \Lambda_n$ be such that

$$E_n(f) = \|P - f\|.$$

The theorem will follow once we prove that $P \notin \Lambda_{n-1}$.

We note first that if $E_n(f) = 0$, then (1.3) follows immediately, since, by assumption, $f \notin \Lambda_{n-1}$. Hence, we may assume that $E_n(f) > 0$.

Since $(u_0, ..., u_n)$ is a T-system, the well-known characterization of best approximants (see, e.g., [3]) provides for the existence of an $n + 2$-term "alternance" of $f - P$, i.e., of $n + 2$ points $0 \leqslant x_1 < x_2 < \cdots < x_{n+2} \leqslant 1$, such that

$$f(x_i) - P(x_i) = \sigma(-1)^i E_n(f) \qquad i = 1, 2, ..., n + 2, \tag{1.4}$$

where σ is 1 or -1.

We now take one of the sequences $\{x_i\}_1^{n+1}$, $\{x_i\}_2^{n+2}$ for which the value $-E_n(f)$ is attained at the last point. Renaming the selected sequence $\{y_i\}_1^{n+1} (y_1 < y_2 \cdots < y_{n+1})$, we have

$$f(y_{n+1-j}) - P(y_{n+1-j}) = (-1)^{j+1} E_n(f) \qquad j = 0, 1, ..., n. \tag{1.5}$$

Assuming now that $P \in \Lambda_{n-1}$, we also have $f - P \in C(u_0, ..., u_{n-1})$, i.e.,

$$\begin{vmatrix} u_0(y_1) & \cdots & u_0(y_{n+1}) \\ \vdots & & \\ u_{n-1}(y_1) & \cdots & u_{n-1}(y_{n+1}) \\ f(y_1) - P(y_1) & \cdots & f(y_{n+1}) - P(y_{n+1}) \end{vmatrix} \geqslant 0. \tag{1.6}$$

On the other hand, substituting from (1.5) and expanding by the last row, we see that the determinant is negative, since the elements of the last row are nonzero and of alternating signs, and all corresponding minors are positive by (1.1). This contradiction proves the theorem.

THEOREM 2. *Let f belong to $C(u_0, ..., u_{n-1}) \setminus \Lambda_{n-1}$. Then the maximal length of an "alternance" of $f - T_{n-1}(f)$ is $n + 1$.*

Proof. Assuming that there is an $n + 2$-term "alternance" $\{x_i\}_1^{n+2}$, we may repeat the selection process used in the proof of Theorem 1 and arrive at a contradiction.

THEOREM 3. *Let f be a function belonging to $C(u_0, ..., u_{n-1})$, and let $P = \sum_{i=0}^{n} a_i u_i = T_n(f)$; then $a_n \geqslant 0$. If, further, $f \notin \Lambda_{n-1}$, then $a_n > 0$.*

Proof. If $f \in \Lambda_n$, then clearly $f \equiv P$, and since, for all

$$0 \leqslant x_1 < \cdots < x_{n+1} \leqslant 1,$$

$$0 \leqslant \begin{vmatrix} u_0(x_1) & \cdots & u_0(x_{n+1}) \\ \vdots & & \vdots \\ u_{n-1}(x_1) & \cdots & u_{n-1}(x_{n+1}) \\ P(x_1) & \cdots & P(x_{n+1}) \end{vmatrix} = a_n \begin{vmatrix} u_0(x_1) & \cdots & u_0(x_{n+1}) \\ \vdots & & \vdots \\ u_{n-1}(x_1) & \cdots & u_{n-1}(x_{n+1}) \\ u_n(x_1) & \cdots & u_n(x_{n+1}) \end{vmatrix},$$

it follows that $a_n \geqslant 0$. Furthermore, if $f \notin \Lambda_{n-1}$, then obviously $a_n > 0$.

Assuming next that $f \notin \Lambda_n$, it follows that $E_n(f) > 0$ and we may proceed as in the proof of the previous theorem, securing a sequence $\{y_i\}^{n+1}$ for which (1.6) holds. Assuming that $a_n \leqslant 0$, we have $f - P \in C(u_0, ..., u_{n-1})$. Hence a contradiction is reached in the same way as in the proof of Theorem 1.

Combining both parts, the theorem follows.

A similar method establishes also

THEOREM 4. *If f is an n-times continuously differentiable function of $C(u_0, ..., u_{n-1})$, and $\{u_i\}_0^n$ is an Extended Complete Tchebycheff system constructed on $\{w_i\}_0^n$ (for the relevant definitions and properties see* [8] *or* [5]), *then*

$$a_n > \min_t \frac{D_{n-1} \cdots D_0 f(t)}{w_n(t)}, \tag{1.7}$$

where $D_i g = d[g(t)/w_i(t)]/dt$, $i = 0, 1, ..., n$.

In particular if $u_i(t) \equiv t^i/i!$, we have

$$a_n > \min f^{(n)}(t). \tag{1.8}$$

For generalized absolutely monotone functions (see [6] and [2]) we easily derive from Theorem 4 the following

THEOREM 5. *Let f be a generalized absolutely monotone function on* (0, 1) *and let, for $k \geqslant 0$,*

$$T_k(f) = P_k = \sum_{i=0}^{k} a_i u_i .$$

Then

$$\frac{D_{k-1} \cdots D_0 f(0^+)}{w_k(0)} \leqslant a_k, \qquad k = 0, 1, ... \tag{1.9}$$

In particular, if $f(t) \equiv \sum_{k=0}^{\infty} b_k t^k/k!$, with $b_k \geqslant 0$ for all k, then

$$b_k \leqslant a_k \qquad k = 0, 1, ... \tag{1.10}$$

For the special case of ordinary convexity of order n we have, further, the following:

THEOREM 6. *Let $f \in C(1, x, ..., x^{n-1}) \setminus \Lambda(1, x, ..., x^{n-1})$ and set*

$$g(t) \equiv f(t) - T_{n-1}(f)(t).$$

Then

(a) *Both end points of* [0, 1] *are extremal points for* g; *explicitly,*

$$g(1) = (-1)^{n-1} g(0) = \| g \| = E_n .$$

(b) *For* $n \geqslant 2$, *the only* $(+)$ *point greater than the last* $(-)$ *point is* $t = 1$. *Similarly, the only* $(-1)^{n-1}$ *point smaller than the first* $(-1)^n$ *point is* $t = 0$.

Proof. We prove the theorem only for $t = 1$, the proof for the other end point being identical.

By the characterization theorem for best approximants, there exists an $n + 1$-term "alternance". By using an argument similar to that of the proof of Theorem 1, we conclude that in such a sequence of $n + 1$ points the last one must be a $(+)$ point.

Let t^* be the last $(-)$ point. Then there exist points $t_1 < \cdots t_{n-1} < t^* < s$ which are $(-)$ and $(+)$ points alternatingly. We shall show that $s = 1$.

Indeed, assume $s < 1$. Since $g \in C(1, x,..., x^{n-1})$, we have, for the sequence $t_2 < t_3 < \cdots < t^* < s$, the inequality

$$\begin{vmatrix} 1 & \cdots & 1 & 1 & 1 \\ t_2 & \cdots & t^* & s & 1 \\ & & \vdots & \vdots & \vdots \\ (t_2)^{n-1} & \cdots & (t^*)^{n-1} & s^{n-1} & 1 \\ (-1)^{n-1} g & \cdots & -\| g \| & \| g \| & g(1) \end{vmatrix} \geqslant 0.$$

By subtracting $\| g \|$ times the first row from the last and expanding by the last row, we have

$$(g(1) - \| g \|)\, V(t_2 ,..., t^*, s) \geqslant 2 \| g \| \sum_{j=1}^{[n/2]} V(t_2 ,..., t_{n-2j} , t_{n-2j+2} ,..., s, 1), \tag{1.11}$$

where $V(\cdot, \cdot,..., \cdot)$ is the Vandermonde of its arguments. Inequality (1.11) can hold only if $g(1) = \| g \|$ and

$$V(t_1 ,..., t_{n-2j} , t_{n-2j+2} ,..., s, 1) = 0 \qquad \text{for all } j.$$

Since the arguments are distinct, this is clearly impossible. Q.E.D.

Theorem 6 is not valid for general T-systems, as evidenced by the following example:

Consider the interval $[-1/5, 1]$, and let $u_0(t) \equiv 2 - t^2 > 0$, $f(t) \equiv 3t^3$. Since $(d/dt)[f(t)/u_0(t)] = 3[(6t^2 - t^4)/(2 - t^2)^2] > 0$ on $[-1/5, 1]$, $f \in C(u_0)$.

It is easy to see that $T_0(f) = u_0(t)$. Indeed, $f(t) - u_0(t) \equiv 3t^3 + t^2 - 2$ decreases on $[-1/5, 0]$ from -1.936 to -2, and increases on $[0, 1]$ from -2 to $+2$.

Hence, 0 and 1 are the points of alternance, $E_0(f) = 2$, and the left end point is not an extremal point.

II. Nonexistence of a Direct Converse

We analyze in this section the question of finding a converse to the theorems of the previous section. We would like to know, for example, whether, for $u_i = t^i$, $i = 0, 1,\ldots$, the inequalities

$$E_{n-1}([0, 1]; f) > E_n([0, 1]; f) \qquad n = 1, 2,\ldots \tag{2.1}$$

imply that all derivatives of f have constant signs (not necessarily the same) on $(0, 1)$. The answer turns out to be negative; a simple counterexample: take $f(x) \equiv e^x - e^\theta x$ $(\theta > 0)$. The following observation shows that (2.1) tells us very little about f:

Theorem 7. *Let A be the set of functions for which $E_{n-1}([0, 1]; f) > E_n([0, 1]; f)$ for all n. Then A^c, the complement of A, is a set of the first category in $C[0, 1]$.*

Proof. Note that

$$A^c = \bigcup_{n=1}^{\infty} (f; E_{n-1}([0, 1]; f) = E_n([0, 1]; f)).$$

Let

$$\mathscr{B}_n = (f; E_{n-1}([0, 1]; f) = E_n([0, 1]; f)).$$

We shall prove that $\mathscr{B}_n$ has an empty interior.

Let f_0 belong to $\mathscr{B}_n$, and let $Q \in \Lambda_{n-1}$ be such that

$$\| f_0 - Q \| = E_{n-1}(f_0) = E_n(f_0).$$

There exist then $n + 2$-alternance points $x_1 < x_2 < \cdots < x_{n+2}$, and with no loss of generality we may assume that x_1 is a $(+)$ point.

Construct a function $P_n(t) \in \Lambda_n$ which has the same sign as $f_0 - Q$ at $x_1,\ldots, x_{n+1}$ [this can be accomplished by prescribing the zeros of $P_n(t)$ at intermediate points].

Given any $\eta > 0$, the function $f(t) = f_0(t) + (\eta/\| P_n \|) P_n(t)$ satisfies $\| f - f_0 \| < \eta$. We now claim that $E_{n-1}(f) > E_n(f)$.

Indeed,

$$T_n(f) = Q + (\eta/\| P_n \|) P_n(t),$$

since the difference $T_n(f) - f$ has $n + 2$ alternance points.

Thus,

$$E_n(f) = \| f_0 - Q \|.$$

On the other hand, $f - Q = f_0 - Q + (\eta/\| P_n \|) P_n$ takes on at $x_1, x_2, \ldots, x_{n+1}$ values which are greater in absolute value than $E_n(f)$ (and are of alternating sign). Let the minimal of these absolute values be K, $K > E_n(f)$. Then de la Vallée Poussin's theorem [3] implies $E_{n-1}(f) \geqslant K$. Q.E.D.

On the other hand, a well-known result (see [4, p. 260]) implies that $D = \{f; f \in C[0, 1]$, the right-hand derivative of f exists and is finite at some $x \in [0, 1]\}$, is of the first category in $C[0, 1]$.

Since if $f \in C(1, x, \ldots, x^{n-1})$, $n \geqslant 2$, the right-hand derivative of f exists everywhere, it follows that the class $C(1, \ldots, x^{n-1})$, $n \geqslant 2$, is of the first category, and hence much smaller than A. Furthermore, there is a simple example of a nonmonotone function for which $E_{n-1}(f) > E_n(f)$, for all n. Thus, we have proved that (2.1) does not imply even that f (or $-f$) belongs to one generalized convexity cone, let alone to an infinite intersection of such cones.

Remark. Suppose

$$E_{n-1}([\alpha, \beta]; f) > E_n([\alpha, \beta]; f)$$

for all n and all rationals α, β. The same arguments show that even this does not imply that f is absolutely monotone. For the converse to be true we need more inequalities.

III. Converse Theorems

We shall prove in this section that properties of the type considered in Theorems 1–3 can be used to provide a characterization of generalized convexity cones.

The converse theorems of this section are easy to establish if f has n continuous derivatives. The standard limit processes do not work, however, and this is the reason for necessity of the following elaborate argumentation.

We start by proving two lemmas.

LEMMA 3.1. *If $f \notin C(u_0, ..., u_{n-1})$ on $[a, b]$, then there exist an interval $[\alpha, \beta] \subset [a, b]$ and $n + 1$ points in $[\alpha, \beta]$, $y_0, y_1, ..., y_n$, such that*

$$T_{n-1}([\alpha, \beta]; f)(y_{n-i}) - f(y_{n-i}) = (-1)^i E_{n-1}([\alpha, \beta]; f) \qquad i = 0, 1, ..., n. \tag{3.1}$$

Proof. Assume that $f \notin C(u_0, ..., u_{n-1})$. Then there exist $n + 1$ points, $a \leqslant z_0 < z_1 < \cdots < z_n \leqslant b$, for which

$$(1) \qquad \begin{vmatrix} u_0(z_0) & \cdots & u_0(z_n) \\ u_1(z_0) & \cdots & u_1(z_n) \\ \vdots & & \\ u_{n-1}(z_0) & \cdots & u_{n-1}(z_n) \\ f(z_0) & \cdots & f(z_n) \end{vmatrix} < 0.$$

Let $P(x)$ be the best approximant from Λ_{n-1} to f on the set $(z_0, z_1, ..., z_n)$. It is well known [3] that such a $P(x)$ exists, is unique, and that $z_0, ..., z_n$ satisfy

$$(P - f)(z_{n-i}) = \sigma(-1)^i \delta_0 \qquad i = 0, 1, ..., n, \tag{3.2}$$

where $\delta_0 > 0$, and σ is 1 or -1.

Note next that since $P \in \Lambda_{n-1}$, we also have

$$\begin{vmatrix} u_0(z_0) & \cdots & u_0(z_n) \\ \vdots & & \vdots \\ u_{n-1}(z_0) & \cdots & u_{n-1}(z_n) \\ (f - P)(z_0) & \cdots & (f - P)(z_n) \end{vmatrix} < 0.$$

Substituting from (3.2) and expanding by the last row, we conclude that $\sigma = 1$.

Define now

C = set of all positive constants δ such that there exist a function $P_\delta \in \Lambda_{n-1}$ and an $(n + 1)$-tuple $z_0^\delta, z_1^\delta, ..., z_n^\delta$ satisfying

$$(P_\delta - f)(z_{n-i}^\delta) = (-1)^i \delta, \qquad i = 0, 1, ..., n. \tag{3.3}$$

Observe that C is a bounded set. In fact, $C \subset (0, \|f\|]$, since for $\delta > \|f\|$, (3.3) implies that P_δ has to change sign n times and therefore to have at least n zeros. Since $P_\delta \in \Lambda_{n-1}$, this is impossible.

Let now $C_1 = C \cap [\delta_0, \|f\|]$. Then C_1 is a bounded nonempty set, since $\delta_0 \in C_1$. We next show that it is closed.

Let $\{\delta_k\}_{k=1}^\infty$ be a convergent sequence in C_1, and let $\bar{\delta}$ be its limit. We know that $\bar{\delta} \in [\delta_0, \|f\|]$.

Let $\zeta_k = \{z_{k0}, z_{k1}, \ldots, z_{kn}\}$, $k = 1, 2, \ldots$, be the corresponding $(n + 1)$-tuples whose existence is assured by (3.3). Considering these as points in E^{n+1}—the $(n + 1)$-dimensional Euclidean space—and noting that $a \leqslant z_{ij} \leqslant b$ for all i and j, we conclude that $\{\zeta_k\}_{k=1}^{\infty}$ has a convergent subsequence.

Note next that, for all k, i,

$$|(P_k - f)(z_{k,i}) - (P_k - f)(z_{k,i+1})| = 2\delta_k \geqslant 2\delta_0, \tag{3.4}$$

where P_k is the function of Λ_{n-1} corresponding to $\{z_{k0}, \ldots, z_{kn}\}$. Hence the limit of the convergent subsequence is an $(n + 1)$-tuple $(\bar{z}_0, \bar{z}_1, \ldots, \bar{z}_n)$ with distinct $\bar{z}_j$'s. Observe finally that considering

$$P(z_{n-i}) + (-1)^{i+1}\delta = \sum_{j=0}^{n-1} a_j u_j(z_{n-i}) + (-1)^{i+1}\delta = f(z_{n-i}), \qquad i = 0, 1, \ldots, n$$

as a linear system of $n + 1$ equations in the $n + 1$ unknowns $a_1, \ldots, a_n$, δ, it follows that δ, as well as the corresponding polynomial, are continuous functions of the z_i's. Hence, there exists a polynomial P_δ such that

$$(P_\delta - f)(\bar{z}_{n-i}) = (-1)^i\delta.$$

Thus, C_1 is closed.

Let

$$\delta^* = \max\{\delta;\ \delta \in C_1\} \tag{3.5}$$

and let P_{δ^*}, $\{z_0^*, \ldots, z_n^*\}$ be the corresponding polynomial of Λ_{n-1} and the $(n + 1)$-tuple of points, respectively.

Define now

$$\begin{aligned} y_0 &= \max_{z_0^* \leqslant y \leqslant z_1^*} \{y;\ (P_{\delta^*} - f)(y) = (-1)^n \delta^*\}, \\ y_n &= \min_{z_{n-1}^* \leqslant y \leqslant z_n^*} \{y;\ (P_{\delta^*} - f)(y) = \delta^*\}, \\ y_i &= z_i^*, \qquad i = 1, 2, \ldots, n - 1. \end{aligned} \tag{3.6}$$

We claim now that on $[\alpha, \beta] = [y_0, y_n]$,

$$P_{\delta^*} = T_{n-1}([\alpha, \beta]; f). \tag{3.7}$$

Noting the definition of the y_i's, we observe that (3.7) implies (3.1). We assume that

$$\| f - P_{\delta^*} \|_{[\alpha,\beta]} > \delta^*, \tag{3.8}$$

and proceed to show that this violates the definition (3.5) of δ^*.

With no loss of generality we may assume that there exists a point $\bar{y}$ in $[\alpha, \beta]$ such that

$$P_{\delta^*}(\bar{y}) > f(\bar{y}) + \delta^*. \tag{3.9}$$

We also assume that n is odd. A very similar proof, with only slight modifications, establishes the theorem for n even.

By the construction (3.6) and continuity, we find that $\bar{y} < y_{n-1}$. Hence there exists an r, $1 \leqslant r \leqslant (n-1)/2$, such that

$$y_{n-1-2r} < \bar{y} < y_{n+1-2r}.$$

Let now x_i, $i = 1,\ldots, n-1$, be the y_i's with y_{n-2r}, y_{n+1-2r} excluded, and choose w_i, $i = 1,\ldots, n-1$, satisfying

$$w_1 = x_1, \qquad x_{i-1} < w_i < x_i, \qquad i = 2,\ldots, n-1.$$

Define the function $Q \in \Lambda_{n-1}$ by

$$Q(t) = \begin{vmatrix} u_0(w_1) & \cdots & u_0(w_{n-1}) \; u_0(t) \\ \vdots & & \\ u_{n-2}(w_1) & \cdots & u_{n-2}(w_{n-1}) \; u_{n-2}(t) \\ u_{n-1}(w_1) & \cdots & u_{n-1}(w_{n-1}) \; u_{n-1}(t) \end{vmatrix}.$$

Clearly $Q(t)$ changes sign at the w_i's and only there. Furthermore, at the right end point $Q(t) > 0$. Hence, for a sufficiently small positive η, and for all j, on $(x_j - \eta, x_j + \eta)$ the signs of $Q(t)$ and $(P_{\delta^*} - f)(t)$ are identical. Note also that $Q(\bar{y}) > 0$. Hence, there exists a $\vartheta > 0$ such that

$$\begin{aligned} (P_{\delta^*} - \vartheta Q)(\bar{y}) - f(\bar{y}) &= (P_{\delta^*} - f)(\bar{y}) - \vartheta Q(\bar{y}) > \delta^*, \\ (P_{\delta^*} - \vartheta Q)(y_0) - f(y_0) &= -\delta^*, \\ (P_{\delta^*} - \vartheta Q)(y_{n-2i}) - f(y_{n-2i}) &> \delta^* \qquad i = 0, 1,\ldots, (n-1)/2; \quad i \neq r, \\ (P_{\delta^*} - \vartheta Q)(y_{n-2i-1}) - f(y_{n-2i-1}) &\leqslant -\delta^* \qquad i = 0, 1,\ldots, (n-1)/2. \end{aligned}$$

Choosing $\delta^{**} = \min_i ((P_{\delta^*} - \vartheta Q)(\bar{y} - f(\bar{y}); (P_{\delta^*} - \vartheta Q)(y_{n-2i}) - f(y_{n-2i}),$ $i = 0, 1,\ldots, (n-1)/2)$, and making use of the continuity of all the functions involved, we conclude that there exists a $\bar{\delta}$, $\delta^* < \bar{\delta} < \delta^{**}$, such that $\bar{\delta} \in C_1$, contradicting (3.5). Q.E.D.

LEMMA 3.2. *If $f \notin C(u_0,\ldots, u_{n-1})$ on $[a, b]$, then there exists an interval $[\alpha, \beta] \subset [a, b]$ such that $a_n = a_n([\alpha, \beta]; f) < 0$, where $T_n([\alpha, \beta]; f) = \sum_{i=0}^{n} a_i u_i$.*

Proof. Consider the interval $[y_0, y_n]$ secured in the proof of Lemma 3.1. Starting from $\tilde{y}_n = y_n$ which is a $(+)$ point for $T_{n-1}([y_0, y_n]; f) - f$, we take the largest $(-)$ point smaller than y_n, and call it $\tilde{y}_{n-1}$. We next let $\tilde{y}_{n-2}$ be the largest $(+)$ point preceding $\tilde{y}_{n-1}$, etc.

In this manner we obtain an $n + 1$ "alternance" $(\tilde{y}_0,\ldots, \tilde{y}_n)$ such that, on $[\tilde{y}_0, \tilde{y}_n]$, the maximal length of an "alternance" of $T_{n-1}([y_0, y_n]; f) - f$ is $n + 1$. Therefore, the function

$$T_{n-1}([\tilde{y}_0, \tilde{y}_n]; f) \equiv T_{n-1}([y_0, y_n]; f)|_{[\tilde{y}_0, \tilde{y}_n]}$$

is *not* equal to $T_n([\tilde{y}_0, \tilde{y}_n]; f)$.

Let $[\alpha, \beta] = [\tilde{y}_0, \tilde{y}_n]$, and denote $\tilde{T}_k = T_k([\alpha, \beta]; f)$, $k = n - 1, n$. We have

$$|(\tilde{T}_n - f)(\tilde{y}_{n-i})| \leqslant |(\tilde{T}_{n-1} - f)(\tilde{y}_{n-i})| \qquad i = 0, 1,..., n,$$

since $\tilde{T}_n$ is the best approximant from the wider class Λ_n. The construction of the points implies that

$$(-1)^i[\tilde{T}_n - f)(\tilde{y}_{n-i})] \leqslant (-1)^i(\tilde{T}_{n-1} - f)(\tilde{y}_{n-i}) \qquad i = 0, 1,..., n,$$

or

$$(-1)^i(\tilde{T}_n - \tilde{T}_{n-1})(\tilde{y}_{n-i}) \leqslant 0 \qquad i = 0, 1,..., n. \tag{3.10}$$

Strict inequality must prevail for at least one i, since, otherwise, the functions $\tilde{T}_n$ and $\tilde{T}_{n-1}$ would agree on $n + 1$ points and would therefore be identical, contrary to the definition of the interval.

Expanding the determinant

$$\begin{vmatrix} u_0(\tilde{y}_0) & \cdots & u_0(\tilde{y}_n) \\ \vdots & & \\ u_{n-1}(\tilde{y}_0) & \cdots & u_{n-1}(\tilde{y}_n) \\ (T_n - T_{n-1})(\tilde{y}_0) & \cdots & (T_n - T_{n-1})(\tilde{y}_n) \end{vmatrix}$$

by the last row, we conclude that it is negative. Since its sign is clearly equal to that of a_n, the lemma is proved.

We come now to the first "converse" theorem.

THEOREM 8. *Let f be a continuous function on $[a, b]$. Assume that, for all $[\alpha, \beta]$, $a \leqslant \alpha < \beta \leqslant b$,*

$$\deg T_n([\alpha, \beta]; f) = n. \tag{3.11}$$

Then either f or $-f$ belongs to $C(u_0, u_1,..., u_{n-1}) \setminus \Lambda_{n-1}$ (for all $[\alpha, \beta]$).

Proof. Note first that $f \notin \Lambda_{n-1}$ for all $[\alpha, \beta]$; this is a direct consequence of (3.11).

Assume $f \notin C(u_0,..., u_{n-1})$. By Lemma 3.2, there exists an interval $[\alpha_1, \beta_1]$ such that

$$a_n([\alpha_1, \beta_1]; f) = \text{coefficient of } u_n \text{ in } T_n([\alpha_1, \beta_1]; f) < 0.$$

If, further, $-f \notin C(u_0, ..., u_{n-1})$, then there exists another interval $[\alpha_2, \beta_2]$ such that

$$a_n([\alpha_2, \beta_2]; f) = -a_n([\alpha_2, \beta_2]; -f) > 0.$$

Since T_n is a continuous function of the interval, continuity arguments show the existence of an interval $[\alpha_3, \beta_3]$ such that

$$a_n([\alpha_3, \beta_3]; f) = 0,$$

contradicting (3.11). Hence, either f or $-f$ must belong to $C(u_0, ..., u_{n-1})$.

Similarly, Lemma 3.2 implies the following:

THEOREM 9. *Let f be a continuous function on $[a, b]$. Assume that for all $[\alpha, \beta]$, $a \leqslant \alpha < \beta \leqslant b$, we have*

$$a_n = a_n([\alpha, \beta]; f) > 0; \tag{3.12}$$

then $f \in C(u_0, ..., u_{n-1}) \setminus \Lambda_{n-1}$ (for all $[\alpha, \beta] \subset [a, b]$).

If a weak inequality holds in (3.12) *for all* $[\alpha, \beta] \subset [a, b]$, *then $f \in C(u_0, ..., u_{n-1})$.*

THEOREM 10. *Let f be a continuous function on $[a, b]$. Assume that, for all $[\alpha, \beta]$, $a \leqslant \alpha < \beta \leqslant b$, the maximal length of an "alternance" of $f - T_{n-1}([\alpha, \beta); f)$ is $n + 1$; then $f \in C(u_0, ..., u_{n-1}) \setminus \Lambda_{n-1}$ (for all such $[\alpha, \beta]$).*

Conclusion

Let f belong to $C[a, b]$. Then the following statements are equivalent:

(a) $E_{n-1}([\alpha, \beta]; f) > E_n(\alpha, \beta]; f)$, for all $[\alpha, \beta]$, $a \leqslant \alpha < \beta \leqslant b$.

(b) $f \in C(u_0, ..., u_{n-1}) \setminus \Lambda_{n-1}$ for all such $[\alpha, \beta]$.

(c) $a_n([\alpha, \beta]; f) > 0$, for all such $[\alpha, \beta]$, where a_n is the "leading coefficient" of the best approximant from Λ_n to f.

(d) For each such $[\alpha, \beta]$, the maximal length of an "alternance" of $f - T_{n-1}([\alpha, \beta]; f)$ is $n + 1$.

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